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ON THE $(\boldsymbol{u}, \boldsymbol{p})$ PROBLEM IN THE THEORY OF ELASTICITY

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A problem of the theory of elasticity is considered for a body with vectors of displacements $\boldsymbol{u}$ *and loads* $\boldsymbol{p}$ *simultaneously defined on one part of the body and with undefined conditions on the remaining part of the body. For a doubly connected domain, where the vectors* $\boldsymbol{u}$ *and* $\boldsymbol{p}$ *are set on one of its boundaries (inner or outer), an iterative method based on reduction of the initial problem to a sequence of mixed problems is justified.*

Key words: *conventionally well-posed problem, doubly connected elastic domain, iterative method of the solution.*

The problem of finding the stress–strain state in a body on the basis of overdetermined conditions on some part of the body surface (with known vectors of displacements $\boldsymbol{u}$ and loads $\boldsymbol{p}$) and undetermined conditions on the other part of the surface was called the $(\boldsymbol{u}, \boldsymbol{p})$ problem [1, 2]. This problem is a conventionally well-posed problem and reduces, in the case of an isotropic elastic domain, to consecutive solution of the Cauchy problem for the Laplace equation [2], which is known to be well-posed in the class of solutions limited in their absolute values [3]. Hence, the $(\boldsymbol{u}, \boldsymbol{p})$ problem is also well-posed in the class of solutions limited in their absolute values. To solve the latter problem, Dveres and Fomin [4] proposed an iterative process yielding a mixed problem at each iteration. Numerical experiments showed that such an algorithm possesses a fairly high resolution and noise immunity. The convergence of consecutive approximations to the solution of the $(\boldsymbol{u}, \boldsymbol{p})$ problem, however, was not proved in [4]. Such a proof is given in the present paper for a doubly connected elastic domain with the vectors $\boldsymbol{u}$ and $\boldsymbol{p}$ being set on its inner (outer) boundary and undefined conditions on the outer (inner) boundary.

1. Formulation of the Problem. Let us consider an elastic body, which occupies a doubly connected spatial domain v with inner and outer boundaries (S_1 and S_2, respectively) satisfying necessary smoothness conditions [5]; this domain is assumed to obey Hooke's law

$$\varepsilon_{kl} = a_{klmn}\sigma_{mn}, \qquad \sigma_{kl} = b_{klmn}\varepsilon_{mn}. \tag{1.1}$$

Hereinafter, ε_{kl}, σ_{kl}, a_{klmn}, and b_{klmn} are the components of strain tensors, stress tensors, elastic compliances, and elastic moduli, respectively; the subscripts k and l acquire the values 1, 2, and 3; summation is performed over repeated subscripts k and l.

The strains ε_{kl} are expressed in terms of the components u_k of the displacement vector $\boldsymbol{u}$ by the Cauchy relations

$$\varepsilon_{kl} = (1/2)(u_{k,l} + u_{l,k}), \tag{1.2}$$

where the subscript after the comma indicates a partial derivative with respect to the corresponding coordinate.

There are no bulk forces, and the equilibrium equations have the form

$$\sigma_{kl,l} = 0. \tag{1.3}$$

The displacements and loads are known on one boundary of the domain v, e.g., on the boundary S_1, i.e.,

$$\boldsymbol{u} = \boldsymbol{u}_*, \qquad \boldsymbol{p} = \boldsymbol{p}_* \qquad \text{on} \quad S_1, \tag{1.4}$$

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where $\boldsymbol{p} = \{p_k\}$, $p_k = \sigma_{kl} n_l$ and n_k are the components of a unit vector of the external normal to S_1, and $\boldsymbol{u}_* = \{u_{k*}\}$ and $\boldsymbol{p}_* = \{p_{k*}\}$ are functions defined on S_1. We assume that $u_{k*} \in H^{1/2}(S_1)$ and $p_{k*} \in H^{-1/2}(S_1)$ (the spaces used here and in what follows are defined in [5]).

It should be noted that problem (1.1)–(1.4) formulated for determining the stress–strain state in the domain v arises also in considering a linearly elastic (viscoelastic) domain containing a physically nonlinear inclusion, where one has to generate a necessary (e.g., homogeneous) stress–strain state by choosing appropriate loads on the outer boundary of the domain [6, 7] or to obtain a necessary current or final shape of the inclusion, i.e., the corresponding displacements of the points of its boundary [8].

2. Iterative Method of Solving Problem (1.1)–(1.4). To solve the $(\boldsymbol{u}, \boldsymbol{p})$ problem [for a simply connected domain v with the boundary $S = S_1 \cup S_2$ with conditions (1.4) being set on some part S_1 of this boundary], Dveres and Fomin [4] proposed and tested the following iterative process. At the zeroth iteration ($n = 0$), it is assumed that $\boldsymbol{p} = \boldsymbol{p}_*$ on S_1 and $\boldsymbol{u} = \boldsymbol{u}^0$ on S_2 ($\boldsymbol{u}^0$ is an arbitrary piecewise-continuous function, e.g., $\boldsymbol{u}^0 = 0$). Solving this mixed problem, one can determine the stress–strain state in the domain v and the vectors $\boldsymbol{u}^0$ and $\boldsymbol{p}^0$ on the entire boundary $S = S_1 \cup S_2$. After that, the first condition in (1.4) is chosen on the part S_1 of the boundary at odd iterations and the second condition in (1.4) is chosen at even iterations; the loads and displacements found at the previous iteration are used on the part S_2 of the boundary. Thus, the boundary conditions have the form

$$\begin{gathered} \boldsymbol{u}^{2n-1} = \boldsymbol{u}_* \quad \text{on} \quad S_1, \qquad \boldsymbol{p}^{2n-1} = \boldsymbol{p}^{2n-2} \quad \text{on} \quad S_2, \\ \boldsymbol{p}^{2n} = \boldsymbol{p}_* \quad \text{on} \quad S_1, \qquad \boldsymbol{u}^{2n} = \boldsymbol{u}^{2n-1} \quad \text{on} \quad S_2 \qquad (n = 1, 2\ \ldots). \end{gathered} \tag{2.1}$$

As was noted above, the convergence of this iterative process to the solution of the $(\boldsymbol{u}, \boldsymbol{p})$ problem was not proved in [4].

Let us demonstrate that the sequence of the solutions $\boldsymbol{u}^n$ of the mixed problems (1.1)–(1.3), (2.1) for the doubly connected domain considered reduces to the solution of the initial problem (1.1)–(1.4). We need to clarify that the vector $\boldsymbol{u}^0 = \{u_k^0\}$ on S_2 at the zeroth iteration is not arbitrary but is chosen in a manner that $u_k^0 \in H^{1/2}(S_2)$. Therefore, the solution $\boldsymbol{u}^0$ in the domain v of the mixed problem exists, and $u_k^0 \in H^1(v)$ [5]. Then, with allowance for the above-made assumptions about the functions $\boldsymbol{u}_*$ and $\boldsymbol{p}_*$ in Eq. (1.4), it follows from Eq. (2.1) that $u_k^n \in H^1(v)$ for the displacement vector $\boldsymbol{u}^n = \{u_k^n\}$ at each iteration, because the condition $u_k^n \in H^{1/2}(S_2)$ or $p_k^n \in H^{-1/2}(S_2)$ is satisfied on S_2.

We introduce the norm for the field of displacements

$$\|\boldsymbol{u}\| = \Big(\int_v b_{klmn} u_{k,l} u_{m,n} \, dv \Big)^{1/2},$$

which is equivalent to the norm $\|\boldsymbol{u}\|_{H^1(v)}$ [5, 8]. Because of (1.1)–(1.3) and the known equation of virtual work, we obtain the equality

$$\|\boldsymbol{u}\|^2 = \int_v \varepsilon_{kl} \sigma_{kl} \, dv = \int_S \boldsymbol{u} \cdot \boldsymbol{p} \, dS \qquad (S = S_1 \cup S_2). \tag{2.2}$$

We consider the numerical sequence $\{a_n\}$ ($a_n = \|\Delta \boldsymbol{u}^n\|^2$, where $\Delta \boldsymbol{u}^n = \boldsymbol{u}^{n+1} - \boldsymbol{u}^n$ and $n = 1, 2, \ldots$). Based on Eq. (2.2) and the equality $\Delta \boldsymbol{u}^n \cdot \Delta \boldsymbol{p}^n|_{S_2} = 0$ following from Eq. (2.1), the general term of this numerical sequence can be presented as

$$a_n = \int_{S_1} \Delta \boldsymbol{u}^n \cdot \Delta \boldsymbol{p}^n \, dS \geq 0 \qquad (n = 1, 2, \ldots). \tag{2.3}$$

From Eqs. (2.1) and (2.3), we find

$$\begin{gathered} a_{2n-1} = \int_{S_1} (\boldsymbol{p}_* - \boldsymbol{p}^{2n-1}) \cdot (\boldsymbol{u}^{2n} - \boldsymbol{u}_*) \, dS, \qquad a_{2n} = \int_{S_1} (\boldsymbol{p}_* - \boldsymbol{p}^{2n+1}) \cdot (\boldsymbol{u}^{2n} - \boldsymbol{u}_*) \, dS, \\ a_{2n+1} = \int_{S_1} (\boldsymbol{p}_* - \boldsymbol{p}^{2n+1}) \cdot (\boldsymbol{u}^{2n+2} - \boldsymbol{u}_*) \, dS. \end{gathered} \tag{2.4}$$

Herefrom, we obtain

$$a_{2n} - a_{2n-1} = \int\limits_{S_1} (\boldsymbol{p}^{2n-1} - \boldsymbol{p}^{2n+1}) \cdot (\boldsymbol{u}^{2n} - \boldsymbol{u}_*)\, dS = \int\limits_{S} (\boldsymbol{p}^{2n-1} - \boldsymbol{p}^{2n+1}) \cdot (\boldsymbol{u}^{2n} - \boldsymbol{u}^{2n-1})\, dS$$

because $\boldsymbol{u}^{2n-1} = \boldsymbol{u}_*$ on S_1 and $\boldsymbol{u}^{2n-1} = \boldsymbol{u}^{2n}$ on S_2. Owing to Betti's identity and the equalities $\boldsymbol{u}^{2n-1} = \boldsymbol{u}^{2n+1} = \boldsymbol{u}_*$ on S_1 and $\boldsymbol{p}^{2n} = \boldsymbol{p}^{2n+1}$ on S_2, we obtain

$$a_{2n} - a_{2n-1} = \int\limits_{S} (\boldsymbol{u}^{2n-1} - \boldsymbol{u}^{2n+1}) \cdot (\boldsymbol{p}^{2n} - \boldsymbol{p}^{2n-1})\, dS = -\|\boldsymbol{u}^{2n-1} - \boldsymbol{u}^{2n+1}\|^2.$$

Similarly to the use of Eq. (2.4) and the equalities $\boldsymbol{p}^{2n} = \boldsymbol{p}^{2n+2} = \boldsymbol{p}_*$ on S_1 and $\boldsymbol{p}^{2n} = \boldsymbol{p}^{2n+1}$ and $\boldsymbol{u}^{2n+1} = \boldsymbol{u}^{2n+2}$ on S_2, we obtain

$$a_{2n+1} - a_{2n} = \int\limits_{S_1} (\boldsymbol{p}_* - \boldsymbol{p}^{2n+1}) \cdot (\boldsymbol{u}^{2n+2} - \boldsymbol{u}^{2n})\, dS = \int\limits_{S} (\boldsymbol{p}^{2n} - \boldsymbol{p}^{2n+1}) \cdot (\boldsymbol{u}^{2n+2} - \boldsymbol{u}^{2n})\, dS$$

$$= \int\limits_{S} (\boldsymbol{u}^{2n} - \boldsymbol{u}^{2n+1}) \cdot (\boldsymbol{p}^{2n+2} - \boldsymbol{p}^{2n})\, dS = -\|\boldsymbol{u}^{2n} - \boldsymbol{u}^{2n+2}\|^2.$$

Thus, for all n, we have

$$a_n = a_{n-1} - \|\boldsymbol{u}^{n+1} - \boldsymbol{u}^{n-1}\|^2. \tag{2.5}$$

It follows from Eq. (2.5) that sequence (2.3) is decreasing and bounded from below ($a_n \geq 0$). Therefore, there exists $\lim\limits_{n\to\infty} a_n \geq 0$. Then, from Eq. (2.5) we obtain $\lim\limits_{n\to\infty} \|\boldsymbol{u}^{n+1} - \boldsymbol{u}^{n-1}\| = 0$, i.e., $\boldsymbol{u}^{n+1} \to \boldsymbol{u}^{n-1}$ in v and $\boldsymbol{u}^{n+1} \to \boldsymbol{u}^{n-1}$ and $\boldsymbol{p}^{n+1} \to \boldsymbol{p}^{n-1}$ on S.

By virtue of Eq. (2.1), on the part S_2 of the boundary, we have $\boldsymbol{u}^{2n+1} \to \boldsymbol{u}^{2n-1} = \boldsymbol{u}^{2n}$ and $\boldsymbol{p}^{2n+1} = \boldsymbol{p}^{2n}$ at odd iterations and $\boldsymbol{u}^{2n} = \boldsymbol{u}^{2n-1}$ and $\boldsymbol{p}^{2n} = \boldsymbol{p}^{2n+1} \to \boldsymbol{p}^{2n-1}$ at even iterations.

Thus, $\Delta\boldsymbol{u}^n = \boldsymbol{u}^{n+1} - \boldsymbol{u}^n \to 0$ and $\Delta\boldsymbol{p}^n = \boldsymbol{p}^{n+1} - \boldsymbol{p}^n \to 0$ on S_2 as $n \to \infty$.

Considering the $(\Delta\boldsymbol{u}^n, \Delta\boldsymbol{p}^n)$ problem for the domain v with the vectors $\Delta\boldsymbol{u}^n$ and $\Delta\boldsymbol{p}^n$ being known at the outer boundary S_2 of this domain [from the solution of problem (1.1)–(1.3), (2.1)], with $\Delta u_k^n \in H^{1/2}(S_2)$ and $\Delta p_k^n \in H^{-1/2}(S_2)$, and with strains $\Delta\varepsilon_{kl}^n$ and stresses $\Delta\sigma_{kl}^n$ being related through Hooke's law (1.1), we conclude that $\Delta u_k^n \in H^1(v) \to 0$ because $\Delta u_k^n \to 0$ and $\Delta p_k^n \to 0$ on S_2. Hence, $\Delta\boldsymbol{p}^n \to 0$ and $\Delta\boldsymbol{u}^n \to 0$ on S_1, i.e., $\boldsymbol{p}^n \to \boldsymbol{p}_*$ and $\boldsymbol{u}^n \to \boldsymbol{u}_*$, which was to be proved.

It should also be noted that, if the equality $a_n = a_{n-1}$ holds for some value of n, i.e., $\boldsymbol{u}^{n+1} = \boldsymbol{u}^{n-1}$ in v in accordance with Eq. (2.5), this means that the exact solution of the $(\boldsymbol{u}, \boldsymbol{p})$ problem is obtained already at the zeroth iteration, i.e., the value of the displacement vector $\boldsymbol{u}^0$ on S_2 is correctly "guessed." Indeed, let, e.g., $\boldsymbol{u}^{2n+1} = \boldsymbol{u}^{2n-1}$ in v. Then, from Eq. (2.1), we obtain $\boldsymbol{p} = \boldsymbol{p}^{2n-1} = \boldsymbol{p}^{2n-2}$ and $\boldsymbol{u} = \boldsymbol{u}^{2n-1}$ on the boundary S_2 at the $(2n-1)$th iteration, $\boldsymbol{u} = \boldsymbol{u}^{2n} = \boldsymbol{u}^{2n-1}$ and $\boldsymbol{p} = \boldsymbol{p}^{2n}$ at the $(2n)$th iteration, $\boldsymbol{u} = \boldsymbol{u}^{2n+1} = \boldsymbol{u}^{2n-1}$ and $\boldsymbol{p} = \boldsymbol{p}^{2n+1} = \boldsymbol{p}^{2n-1} = \boldsymbol{p}^{2n}$ at the $(2n+1)$th iteration, i.e., we have $\boldsymbol{u} = \boldsymbol{u}^{2n-1}$ and $\boldsymbol{p} = \boldsymbol{p}^{2n-1}$ on S_2 at even and odd iterations. By virtue of the uniqueness of the solution of the $(\boldsymbol{u}, \boldsymbol{p})$ problem (for the vectors $\boldsymbol{u}$ and $\boldsymbol{p}$ being set on S_2), we obtain $\boldsymbol{p}^{2n-1} = \boldsymbol{p}_*$ and $\boldsymbol{u}^{2n-1} = \boldsymbol{u}_*$ on S_1; the found values of $\boldsymbol{u}^{2n-1}$ and $\boldsymbol{p}^{2n-1}$ on S_2 correspond to the exact solution. As in all previous (and subsequent) approximations, these quantities ($\boldsymbol{u}^{2n-1}$ or $\boldsymbol{p}^{2n-1}$) are contained in the boundary conditions on S_2 (e.g., $\boldsymbol{p}^{2n-2} = \boldsymbol{p}^{2n-1}$ on S_2), then we obtain the exact solution of this problem, including that for $n = 0$, by going "upward" through iterations.

We can also note that, if the boundary conditions on S_1 are supplemented by similar conditions on S_2, i.e.,

$$\begin{aligned} \boldsymbol{u}^{2n-1} &= \boldsymbol{u}_* \quad \text{on} \quad S_1, \qquad \boldsymbol{u}^{2n-1} = \boldsymbol{u}^{2n-2} \quad \text{on} \quad S_2, \\ \boldsymbol{p}^{2n} &= \boldsymbol{p}_* \quad \text{on} \quad S_1, \qquad \boldsymbol{p}^{2n} = \boldsymbol{p}^{2n-1} \quad \text{on} \quad S_2 \end{aligned} \tag{2.6}$$

is used instead of Eq. (2.1), the iterative process becomes divergent. (This fact was noted in [4], but no proof was given.) Indeed, Eqs. (2.4) are also valid for the sequence $a_n = \|\Delta\boldsymbol{u}^n\|^2$, but it follows in this case from Eq. (2.6) and Betti's identity that the equality $a_n = a_{n-1} + \|\boldsymbol{u}^{n+1} - \boldsymbol{u}^{n-1}\|^2$ holds for all n, i.e., the sign in the right side of Eq. (2.5) changes to the opposite one, and the sequence $\{a_n\}$ becomes increasing.

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